Assignment 3

(20 April, 2026)

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Course: Algebraic Topology II (KSM4E02) Instructor: Aritra Bhowmick

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Q1. (*Ext Functors and (co)Product*) Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be R-modules. Prove the following isomorphisms.

1. $\operatorname{Ext}^n_R\left(\bigoplus_{i\in I} A_i, B\right) = \prod_{i\in I} \operatorname{Ext}^n_R(A_i, B)$, for R-module B.
2. $\operatorname{Ext}^n_R\left(A, \prod_{j\in J} B_j\right) = \prod_{j\in J} \operatorname{Ext}^n_R(A, B_j)$, for R-module A.

$5 + 5 = 10$

Q2. (*Interval Chain Complex*) Given a ring R, consider the *inteval chain complex*

$$I := \left(\cdots \to 0 \to \underset{(1)}{R\langle e\rangle} \xrightarrow{e\mapsto v_1 - v_0} \underset{(0)}{R\langle v_0, v_1\rangle} \to 0 \to \cdots \right).$$

Note that I can be thought of as the *cellular chain complex* of $[0,1]$ with the standard CW structure of two 0 -cells and one 1-cell.

a. Describe $C\otimes I$ explicitly, and define two natural chain maps $\iota_0, \iota_1 : C \hookrightarrow C\otimes I$.
b. Given two chain maps $f, g : C \to D$, show that a chain homotopy $h : f \simeq g$ can be naturally identified as a *chain map* $H : C\otimes I \to D$ such that $H\circ\iota_0 = f, H\circ\iota_1 = g$.
c. Describe explicitly the *cone* on a chain complex A defined as $\operatorname{Cone}(A) := \operatorname{coker}\left(A \xrightarrow{\iota_1} A\otimes I\right)$.

   Hint : (co)kernels in the category of chain complexex is computed degreewise, with induced boundary maps.
d. Verify that for any complex A, we have $\operatorname{Cone}(A)$ is acyclic, i.e, it has vanishing homology.
e. Verify that the *suspension* of A, defined as $\Sigma A := \operatorname{coker}\left(A \xrightarrow{\iota_0} \operatorname{Cone}(A)\right)$, is nothing but the shifted complex $A[-1]$.

These exercises are building blocks to do *homotopy theory* on the category of chain complexes. Note that one can also define the cone isomorphically as $\operatorname{Cone}(A)_n = A_n \oplus A_{n-1}$ with boundary $\begin{pmatrix} \partial & \mathrm{Id} \\ 0 & -\partial \end{pmatrix}$.

$2 \times 5 = 10$

Q3. Consider the space $X_g = \underbrace{S^1 \vee \ldots \vee S^1}_{g\text{-times}} \vee S^2$.

a. Compute the homology with $\mathbb{Z}$ coefficients of X_g.
b. Compute the cohomology with G coefficients of X_g, for any $\mathbb{Z}$-mod G.
c. Verify that X_g has the same (co)homology groups as $\Sigma_g = \underbrace{\mathbb{T}\#\cdots\#\mathbb{T}}_{g\text{-times}}$, with any coefficient.
d. Justify that X_g is not homotopy equivalent to Σ_g.

$2 + 2 + 2 + 4 = 10$

Q4. (*Bockstein Homomorphism*) Given a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, and a chain complex $S_\bullet$ of free R-modules, we have the short exact sequence of chain complexes

$$0 \to \hom(S_\bullet, A) \to \hom(S_\bullet, B) \to \hom(S_\bullet, C) \to 0.$$

Passing to the long exact sequence in cohomology, we have the boundary map $\beta : H^n(\hom(S_\bullet, C)) \to H^{n+1}(\hom(S_\bullet, A))$. In particular, taking the singular chain complex $S_\bullet(X)$ of a space, we get a map $\beta : H^n(X; C) \to H^{n+1}(X; A)$, which is known as the *Bockstein homomomorphism*.

For some $m > 1$, consider the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z} & \xrightarrow{\bmod m} & \mathbb{Z}_m & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle \bmod m} & & \big\downarrow {\scriptstyle \bmod m^2} & & \big\| & & \\ 0 & \longrightarrow & \mathbb{Z}_m & \xrightarrow[\times m]{} & \mathbb{Z}_{m^2} & \xrightarrow[\bmod m]{} & \mathbb{Z}_m & \longrightarrow & 0 \end{array}$$

Denote the corresponding Bockstein maps $\beta : H^n(X; \mathbb{Z}_m) \to H^{n+1}(X; \mathbb{Z}_m)$, and $\tilde{\beta} : H^n(X; \mathbb{Z}_m) \to H^{n+1}(X; \mathbb{Z})$. Prove the following.

a. There is a commuting diagram

$$\begin{array}{ccccccc} H^n(X;\mathbb{Z}) & \xrightarrow{\rho} & H^n(X;\mathbb{Z}_m) & \xrightarrow{\tilde{\beta}} & H^{n+1}(X;\mathbb{Z}) & \xrightarrow{\eta} & H^{n+1}(X;\mathbb{Z}) \\ & & & \searrow^{\beta} & \big\downarrow {\scriptstyle \rho} & & \\ & & & & H^{n+1}(X;\mathbb{Z}_m) & & \end{array}$$

where the row is exact. Here, ρ is induced from $\mathbb{Z} \xrightarrow{\bmod m} \mathbb{Z}_m$, and η is induced from $\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$.

b. $\beta \circ \beta = 0$.

c. $\beta(a \smile b) = \beta(a) \smile b + (-1)^{|a|} a \smile \beta(b)$, for homogeneous elements $a, b \in H^\star(X; \mathbb{Z}_m)$

In particular, β acts as a *coderivation* on $H^\star(X; \mathbb{Z}_m)$. Moreover, the cohomology of the cochain complex $(H^n(X; \mathbb{Z}_m), \beta)$ is called the *Bockstein cohomology* of X.

$3 + 2 + 5 = 10$

Q5. (*Cohomology Ring of Lens Space*) Recall the lens space L_n constructed as the three-dimensional CW complex as follows. Take n-many 3-simplices $T_1, \ldots, T_n$. First, join them cyclically, and then identify the bottom face of T_i with the top face of T_{i+1}, with indices taken $\bmod n$. Compute the cohomology ring of L_n with $\mathbb{Z}$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ coefficients.

Hint : For n odd, the cohomology ring with $\mathbb{Z}_n$ should be $\frac{\mathbb{Z}_n[X,Y]}{\langle X^2, Y^2 \rangle}$ with $|X| = 1, |Y| = 2$. For n even, the ring should be $\frac{\mathbb{Z}_n[X,Y]}{\langle X^2 - \frac{n}{2}Y, Y^2 \rangle}$ with $|X| = 1, |Y| = 2$. In both cases, XY should correspond to a generator of $H^3(L_n; \mathbb{Z}_n)$. One can identify $Y = \beta(X)$, where $\beta : H^1(L_n; \mathbb{Z}_n) \to H^2(L_n; \mathbb{Z}_n)$ is the Bockstein, which is an isomorphism here.

$2 + 8 = 10$